

# Random planar graphs and beyond

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**Abstract.** We survey several results on the enumeration of planar graphs and on properties of random planar graphs. This includes basic parameters, such as the number of edges and the number of connected components, and extremal parameters such as the size of the largest component, the diameter and the maximum degree. We discuss extensions to graphs on surfaces and to classes of graphs closed under minors. Analytic methods provide very precise results for random planar graphs. The results for general minor-closed classes are less precise but hold with wider generality.

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## 1. Introduction

The theory of random graphs, initiated by Erdős and Rényi [34] in the early 1960s, has become one of the main areas of research in combinatorics [13, 46]. The model studied originally was the class  $G(n, M)$  of graphs with  $n$  labelled vertices and  $M$  edges, equipped with the uniform distribution. Closely related is the binomial model  $G(n, p)$ , in which every possible edge between two vertices is selected independently with probability  $p$ . The two models are very similar if  $p\binom{n}{2}$  is close to  $M$ . The advantage of the  $G(n, p)$  model is the key property of *independence*, which allows to compute probabilities of basic events exactly, and to determine precise thresholds for basic properties such as being acyclic, connected or Hamiltonian. For instance, the probability that three given vertices span a triangle is exactly  $p^3$ , and the probability that a given vertex is isolated is  $(1 - p)^{n-1}$ .

Things become more difficult if we want to analyze random graphs subject to a global condition, such as being regular, planar or triangle-free. Consider the property of being triangle-free: we cannot select edges independently of each other, since once some edges are selected, other edges are forbidden because they would create triangles. How does one proceed in these cases? Simplifying we can say that there are two ways for analyzing random graphs from a constrained class of graphs: either finding a simpler model that is close enough to the class, or counting graphs in the class, or a combination of both. The first method is well exemplified by the class of regular graphs. In the pairing model for  $d$ -regular graphs there are  $n$

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vertices, each of them equipped with  $d$  half-edges. A random pairing of the  $dn$  half-edges produces a random  $d$ -regular multigraph. Probabilities of elementary events can be computed reasonably well, including the probability that the resulting graph is simple. This allows to obtain precise estimates on the number of regular graphs and has led to a rich theory of random regular graphs [74].

Another example is the class of triangle-free graphs. It was proved in [33] that, as the number of vertices goes to infinity, almost all triangle-free graphs are bipartite. Random bipartite graphs can be analyzed in a model very similar to the  $G(n, p)$  model, where again we have independence, and this provides a suitable model for triangle-free graphs. More generally, almost all graphs not containing the complete graph  $K_t$  as a subgraph are  $(t-1)$ -partite [49], and again we have a model similar to  $G(n, p)$ . Even more generally, if  $H$  is a graph with the property that there exists an edge  $e$  such that  $\chi(H-e) < \chi(H)$  (here  $\chi$  denotes the chromatic number) and  $t = \chi(H) \geq 3$ , then almost every graph not containing  $H$  as a subgraph is  $(t-1)$ -partite [67]. These are important examples of monotone classes. A class of graphs is *monotone* if it is closed under taking subgraphs, and it is *hereditary* if it is closed under taking induced subgraphs. Much work has been done on estimating the growth rate of monotone and hereditary classes and on analyzing random graphs from these classes. This is an active area of research closely related to extremal graph theory [14].

The foremost example of the second method for analyzing random graphs, based on counting, is the class of trees. We know how to count trees very precisely (whether labelled or unlabeled, rooted or unrooted) and we also know how to count trees, for instance, with given degrees or with given height. Thus we can analyze random variables like the number of leaves or the height in random trees. Trees are fundamental objects in computer science and powerful methods have been developed for analyzing them. The main tools in this area are generating functions and analytic methods for deriving asymptotic estimates. We enter here the realm of *analytic combinatorics*, as developed by Flajolet and Sedgewick [35]; see also [24] for many aspects of random trees.

The key property that allows us to count trees is that they admit a simple combinatorial decomposition. A rooted tree can be decomposed uniquely into the root and a collection (ordered or not) of subtrees attached to the root. This decomposition translates into equations for the corresponding generating functions, and we are in a situation to apply the methods of analytic combinatorics. Many other combinatorial objects can be decomposed according to simple schemes. This includes the class of planar *maps*. A map is a connected planar multigraph (loops and multiple edges allowed) with a fixed embedding in the plane. In the 1960s Tutte, motivated by the Four Colour Problem, created the theory of map enumeration. He realized that maps admit recursive decompositions, implying algebraic equations for the associated generating functions. He found *exact* formulas for the number of various classes of rooted maps (to be defined later) with given number of edges. For instance, Tutte showed [73] that the number of rooted maps with  $n$  edges equals

$$\frac{2 \cdot 3^n (2n)!}{n!(n+2)!}. \quad (1)$$

This formula and similar ones were later explained more combinatorially, using bijections with certain classes of enriched trees [70]. As we discuss later, these bijections have powerful implications on the structure of random maps.

It took time to realize that the theory of map enumeration could be used to count planar graphs *without* an embedding. This was done first for 2-connected planar graphs by Bender, Gao and Wormald [9], using the enumeration of 3-connected planar maps and Whitney's theorem, namely that a 3-connected planar graph has a unique embedding in the sphere up to homeomorphism. Soon after that the analysis was extended to arbitrary planar graphs by Giménez and Noy [42]. They provided a precise estimate for the number  $G_n$  of planar graphs with  $n$  labelled vertices of the form

$$G_n \sim c n^{-7/2} \gamma^n n!, \quad (2)$$

where  $\gamma \approx 27.2269$  is a well-defined constant, known as the growth constant of planar graphs. This opened the way to the fine analysis of random planar graphs. In the same work [42] it was proved that the number of connected components in a random planar graph follows asymptotically a Poisson distribution plus 1 and that the number of edges is asymptotically Gaussian with linear mean and variance. This is developed in Section 3 for the more basic parameters, and in Section 4 for more advanced extremal parameters, such as the diameter, the maximum vertex degree or the size of the largest block.

The next step was to enumerate graphs that can be embedded in a fixed surface  $S$ , orientable or not. McDiarmid [55] showed first that the growth constant for graphs embeddable in a surface does not depend on the surface (a result already known for maps) and is equal to  $\gamma$ . Soon after that the enumeration of graphs on surfaces was completed independently in [7] and [18]. It was shown that the number of labelled graphs with  $n$  vertices that can be embedded in the orientable surface of genus  $g$  is asymptotically

$$c_g n^{5(g-1)/2-1} \gamma^n n!. \quad (3)$$

We see that only the subexponential term depend on the genus. It is worth remarking that, unlike the planar case, the counting series of graphs in a surface is not computed exactly but rather sandwiched coefficient-wise between two computable series with the same leading asymptotic terms (more details in Section 5). In addition, it was shown [18] that basic parameters, such as the number of components, the number of edges, or the sizes of the largest component and the largest block, have the same asymptotic distribution as for planar graphs. All these results hold as well for graphs on the non-orientable surface of genus  $h$ , in which case the subexponential term in the asymptotics is  $n^{5(h-2)/4-1}$ .

Graphs on surfaces are strongly related to graph minors. A graph  $H$  is a minor of  $G$  if  $H$  can be obtained from a subgraph of  $G$  by contracting edges. A class of graphs is *minor-closed* if it is closed under taking minors. A basic example is the class of planar graphs and, more generally, the class of graphs embeddable in a fixed surface. Other interesting minor-closed classes are series-parallel graphs,  $\Delta Y$ -reducible graphs and graphs with bounded tree-width. Only in a few cases we have access to the counting generating functions, allowing for a precise analysis as in the

case of planar graphs or graphs on surfaces. However one can use combinatorial arguments to prove relevant results on random graphs from a minor-closed class. This program has been carried out mostly by McDiarmid and his coauthors. The results are less precise than those obtained using generating functions and analytic methods, but apply to more general situations. One example can illustrate this: the class  $\mathcal{K}$  of graphs not containing  $K_5$  as a minor is an interesting class (studied by Wagner, motivated by the Four Colour Problem) containing the class of planar graphs. We do not know how to compute the counting generating function for the class  $\mathcal{K}$ , but from the results in [56] it follows that the number of components converges to a Poisson law plus 1 and that the expected number of vertices in the largest component is  $n - c$  for some constant  $c$ . These remarkable results apply to any minor-closed class subject to mild hypothesis, as discussed in Section 6.

There is one general situation where analytic methods still apply, namely when the class of graphs is *subcritical*. This is a technical condition defined in terms of the singularities of the generating functions, but combinatorially it can be interpreted as the fact that the class contains ‘relatively few’ 3-connected graphs. This category includes forests, outerplanar graphs, series-parallel graphs and related classes of graphs. Graphs in these classes have typically a tree-like structure and in fact share several properties with trees. The analysis of subcritical classes uses general tools from analytic combinatorics [25, 44] and will be reviewed in Section 7.

We conclude the paper with some remarks and open problems. In the rest of the paper, unless mentioned otherwise, all graphs are labelled and  $n$  denotes the number of vertices. For the generating functions that will appear, variable  $x$  is associated to vertices and variable  $y$  to edges. For maps,  $n$  denotes the number of edges and  $z$  is the variable associated to edges.

## 2. Planar maps and graphs

Let us go back to Tutte and the enumeration of planar maps. Rooted trees are easier to enumerate than unrooted ones, since the root vertex gives a starting point for the combinatorial decomposition. In the same way Tutte decided to root maps: an edge (not a vertex) is selected and given an orientation. Let  $\mathcal{M}$  be a planar map and let  $e$  be its root edge. Tutte’s analysis distinguished two cases, depending on whether  $\mathcal{M} - e$  is connected or not. In order to keep control of the decomposition, he had to consider the number  $M_{n,k}$  of maps with  $n$  edges in which the root face (the one to the right of the oriented root edge) has degree  $k$ . Analyzing the combinatorial decomposition of maps resulting by removing the root edge, he showed that the generating function  $M(z, u) = \sum_{n,k} M_{n,k} u^k z^n$  satisfies the equation

$$M(z, u) = 1 + zu^2M(z, u)^2 + uz \frac{uM(z, u) - M(z, 1)}{u - 1}. \quad (4)$$

This is a quadratic equation in  $M(z, u)$ , but we cannot solve it directly since it contains the series  $M(z, 1)$ , which is not independent of the unknown. In order

to solve it, Tutte devised what is now known as the *quadratic method*. This is similar to the well-known *kernel method*, but applied to quadratic instead of linear equations. He proved that

$$M(z, 1) = \frac{18z - 1 + (1 - 12z)^{3/2}}{54z^2},$$

and from here the expression in (1) follows easily.

Additional techniques allowed Tutte to enumerate various classes of maps. For instance, in order to count bipartite maps it is enough to restrict the degrees of the faces to be even. Eulerian maps are then enumerated by duality. In his seminal paper [73], Tutte also counted maps according to their connectivity. The unique decomposition of connected graphs into 2-connected and 3-connected components allows us to link the generating functions of maps with given connectivity. If  $B(z) = \sum B_n z^n$  and  $T(z) = \sum T_n z^n$  are, respectively, the generating functions of 2-connected and 3-connected maps (counted according to the number of edges) then the recursive decomposition of a map into its blocks gives

$$M(z) = 1 + B(zM(z)^2). \quad (5)$$

If now  $h(z)$  is the functional inverse of  $(B(z) - 2z)/z$  then the decomposition of a 2-connected map into its 3-connected components gives

$$T(z) = z^2 - \frac{2z^3}{1+z} - zh(z). \quad (6)$$

These equations provide all the information needed. For instance, together with (1), it is easy to derive the asymptotic estimates

$$M_n \sim c_M n^{-5/2} 12^n, \quad B_n \sim c_B n^{-5/2} \left(\frac{27}{4}\right)^n, \quad T_n \sim c_T n^{-5/2} 4^n,$$

for suitable constants  $c_M, c_B, c_T$ .

The enumeration of 3-connected maps is particularly interesting since, by the classical theorem of Steinitz (1922), they correspond precisely to the graphs of convex polytopes in  $\mathbb{R}^3$ . Another reason of interest is that 3-connected planar graphs have a unique embedding in the sphere, a classical result due to Whitney (1933). It follows that there is a one-to-one correspondence between 3-connected planar maps and 3-connected planar graphs. This leads directly to the enumeration of 3-connected labelled planar graphs in which an edge is distinguished and given a direction, corresponding to the root of the associated map. A key feature is that rooted maps have no non-trivial automorphisms, so that all vertices, edges and faces are distinguishable. We can then give them labels and turn a rooted map into a labelled graph. This was first made explicit in [9]. Now, using again Tutte's decomposition of 2-connected graphs into 3-connected components, but in the reverse direction, it is possible to enumerate 2-connected planar graphs. Let us explain how.

In what follows generating functions for graphs are of the exponential type (whereas for maps are ordinary), and variable  $x$  marks vertices and  $y$  marks edges. Let  $T(x, y)$  be the generating function of 3-connected maps, and  $B(x, y)$  that of 2-connected planar graphs. Closely related to  $B(x, y)$  is the generating function  $D(x, y)$  of ‘networks’, which are 2-connected graphs rooted at a directed edge (which may be deleted or not) and whose endpoints are not labelled. Then  $D(x, y)$  is related to  $B(x, y)$  through (see [43] for details)

$$2(1+y)\frac{\partial B}{\partial y}(x, y) = x^2(1+D(x, y)), \quad (7)$$

and  $D(x, y)$  satisfies the equation

$$D(x, y) = (1+y)\exp\left(\frac{x D(x, y)^2}{1+x D(x, y)} + \frac{2}{x^2}\frac{\partial T}{\partial y}(x, D(x, y))\right) - 1. \quad (8)$$

The former two equations are essentially the equivalent of (6) for graphs. They are more involved because there are two variables and several derivatives, but it is just Tutte’s decomposition applied in the reverse direction: from the knowledge of  $T$  we have access to  $D$ , hence to  $B$ . From here it was shown [9] that the number of 2-connected planar graphs grows like

$$c_B n^{-7/2} (\gamma_B)^n n!,$$

where  $\gamma_B \approx 26.18$ . Observe that the polynomial growth is  $n^{-7/2}$  instead of  $n^{-5/2}$ , the reason being that maps are rooted and introduce an extra linear factor. This was a major step since little was known on counting planar graphs, as opposed to the rich theory of counting planar maps created by Tutte and greatly extended later on.

It remained to count connected planar graphs using the decomposition of connected graphs into 2-connected components, and then to count planar graphs in general. Let  $C(x, y)$  and  $G(x, y)$  be the generating functions of connected and arbitrary planar graphs, and let  $C^\bullet(x, y) = x\frac{\partial C}{\partial x}(x, y)$  be that of rooted connected graphs, where a vertex is distinguished as the root. The recursive decomposition of a graph into its blocks implies the equation

$$C^\bullet(x, y) = \exp\left(\frac{\partial B}{\partial x}(x C^\bullet(x, y), y)\right). \quad (9)$$

The former equation is the analog for graphs of (5). And the decomposition of a connected graph into its connected components implies

$$G(x, y) = e^{C(x, y)}, \quad (10)$$

and equation that has no analog for maps since maps are connected by definition. Solving (7), (8) and (9) explicitly is a non-trivial problem. It was done in [42] by finding an explicit expression for  $B(x, y)$  in terms of  $D(x, y)$ ; it is worth remarking that the same solution can be recovered in a more combinatorial way [19]. From

this expression one can determine  $C(x, y)$  as the solution of (9) and then  $G(x, y)$  from (10), thus solving completely the problem of enumerating planar graphs. In particular, the estimate in (2) is obtained. We will not go into the details, which are quite technical, but rather will explain how the solution from [42] opened the way to the fine analysis of random planar graphs.

### 3. Random planar graphs

In addition to the enumerative theory of planar maps, a number of relevant results on random maps were established by Bender, Gao, Richmond and Wormald, among others. A central result in [6] is that a random map almost surely contains linearly many copies of any given planar submap  $M$ . This was later refined by showing that the number of copies of  $M$  is asymptotically normal [40]. These results extend to several classes of maps, such as triangulations and quadrangulations. Another result is that the distribution of vertex degrees follows asymptotically a discrete law with exponential tail; this already follows from Tutte's equations, and later it was shown that the limiting distribution is independent of the surface [37]. A very precise result was obtained for the distribution of the maximum vertex degree [39], proving that it is of order  $\log_{6/5} n$  for maps with  $n$  edges. In another direction, it was shown [38] that a random map contains a unique 2-connected component of linear size, more precisely of size  $n/3$ , a result that extends to more general kind of 'components' in different classes of maps. The limiting distribution of the size of the largest component was obtained in [3], showing that it is non-Gaussian. With respect to metric properties of maps, it was first established in [21] that the typical distance between two vertices in a random quadrangulation is of order  $n^{1/4}$ . As we discuss later this has led to a rich theory of scaling limits of random maps. We will review several of these results when discussing random planar graphs.

The first attempt to analyze random planar *graphs* (without and embedding) was made by in [23]. The probabilistic model is given by the set  $\mathcal{G}_n$  of (labelled) planar graphs with  $n$  vertices equipped with the uniform distribution. The goal declared there was to understand 'what does a random planar graph look like' under this distribution. The authors proved a few preliminary results but it was not until the work of McDiarmid, Steger and Welsh [59] that more significant results were obtained. Among other results, they proved that a random planar graph has with high probability linearly many disjoint *pendant* copies of each fixed connected planar graph  $H$  with a distinguished vertex  $v$ : a pendant copy of  $H$  is a subgraph isomorphic to  $H$  joined to the rest of the graph through a single edge  $uv$ , and such that the isomorphism respects the order of the labels (so that automorphisms are not considered). This implies in particular that there are linearly many vertices of degree  $k$ , for each fixed  $k \geq 1$ . It also implies that a random planar graph has exponentially many automorphisms (consider pendant copies of  $K_{1,2}$  rooted at the vertex of degree two), in sharp contrast with arbitrary random graphs. Another property proved in [59] is that the limiting probability  $p$

that a random planar graph is connected is bounded away from 0 and from 1. In particular, it was proved that  $p \geq e^{-1}$ . At about the same time several authors studied the number of edges in random planar graphs. Using various combinatorial arguments it was proved that almost surely the number of edges is between  $1.85n$  and  $2.44n$ , but no concentration result or limiting distribution was obtained.

The results in [42] allow for a much more precise description. Let  $G_{n,m,k}$  be the number of planar graphs with  $n$  vertices,  $m$  edges, and  $k$  components. The key fact is that it is possible to find an *exact* expression for the exponential generating function

$$G(x, y, u) = \sum_{n,m,k \geq 0} G_{n,m,k} y^m u^k \frac{x^n}{n!}.$$

As we have seen before, exact does not mean simple. However the series  $G(x, y, u)$  can be expressed in terms of the solution of the system of equations (7–10) involving only elementary functions and the generation function  $T(x, y)$  of 3-connected rooted maps counted according to vertices and edges, which is algebraic of degree four. Everything is explicit and computable with the help a computer algebra system; see [43] for a detailed survey.

Let  $X_n$  be the random variable equal to the number of edges in planar graphs with  $n$  vertices. The distribution of  $X_n$  is completely encoded in the generating function  $A(x, y) = G(x, y, 1)$ , since the probability generating function of  $X_n$  is simply

$$p_n(y) = \frac{[x^n]A(x, y)}{[x^n]A(x, 1)},$$

where  $[x^n]$  denotes the coefficient of  $x^n$ . From the system of equations satisfied by  $G$  it is possible to extract, using analytic methods, information on the rate of growth of its coefficients. In this case one proves that, for fixed  $y > 0$ , we have the estimate

$$[x^n]A(x, y) \sim c(y)n^{-7/2}\gamma(y)^n n!.$$

This already gives the estimate (2) with  $c = c(1)$  and  $\gamma = \gamma(1)$ , but it gives more, namely

$$p_n(y) = \frac{c(y)}{c(1)} \left( \frac{\gamma(y)}{\gamma(1)} \right)^n + O\left(\frac{1}{n}\right), \quad (11)$$

where the error term comes from the method of singularity analysis used in deriving the estimates. The probability generating function is close to being an exact power and extensions of the Central Limit Theorem imply a Gaussian limit law for  $X_n$  (see Section IX.5 in [35]). Moreover, from (11) it follows that the expected value  $\mathbb{E}X_n = p'_n(1)$  is asymptotically  $\mu n$ , where  $\mu = (\gamma'(y)/\gamma(1))$ . The function  $\gamma(y)$  is analytic and computable and one obtains  $\mu \approx 2.21$ . A similar computation gives  $\text{Var}(X_n) \approx 0.43n$ . It is also possible to prove a Local Limit Theorem and to show that  $\mathbb{P}(|X_n - \mathbb{E}X_n| > \epsilon n)$  is exponentially small [42]. This gives a very precise picture of the distribution of the number of edges, highly concentrated around  $\mu n$ .

The analysis of the number of components is easier. We have

$$G(x, 1, u) = e^{uC(x)}, \quad (12)$$



where  $C(x)$  is the generating function of connected planar graphs, and the generating function of graphs with exactly  $k$  components is  $C(x)^k/k!$ . One shows that

$$\frac{[x^n] \frac{1}{k!} C(x)^k}{[x^n] G(x)} \sim \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda},$$

where  $\lambda = C(\gamma^{-1}) \approx 0.037$ . It follows that the random variable equal to the number of components in planar graphs with  $n$  vertices converges to  $1 + \text{Po}(\lambda)$ , a Poisson distribution of parameter  $\lambda$ . In particular, the limiting probability of connectedness is  $p = e^{-\lambda} \approx 0.96$ . As will be seen in the next section, the largest component contains almost all vertices: its expected size is  $n - c$ , where  $c$  is a small constant. The small number of vertices not in the largest component accounts for the fact that  $p < 1$ .

Another parameter of interest is the distribution of the vertex degrees. For fixed  $k \geq 1$ , let  $X_{k,n}$  be the number of vertices of degree  $k$  in planar graphs with  $n$  vertices. As mentioned before,  $X_{k,n}$  is linear in  $n$  with high probability. It is natural then to expect that  $\mathbb{E}X_{k,n} \sim p_k n$  as  $n \rightarrow \infty$ . However, much technical work is needed in order to prove this result. It requires a very fine analysis of the generating function  $G^\bullet(x, w)$  of graphs with a distinguished vertex (the root), where  $w$  marks the degree of the root. It is proved in [27] that the  $p_k$  indeed exist and that  $\sum_{k \geq 1} p_k = 1$ . This is equivalent to saying that the probability that a random vertex in a planar graph has degree  $k$  tends to  $p_k$  as  $n \rightarrow \infty$ , and that the degree distribution converges to a discrete law. The explicit expression for the probability generating function is extremely involved but it is computable and one obtains the first values

$$p_1 \approx 0.037, \quad p_2 \approx 0.16, \quad p_3 \approx 0.24, \quad p_4 \approx 0.19, \quad p_5 \approx 0.13, \quad p_6 \approx 0.09.$$

The distribution decays exponentially like  $p_k \sim q^k k^{-1/2}$ , where  $q \approx 0.67$  is an explicit constant, suggesting that the maximum degree is asymptotically  $\log_{q^{-1}}(n)$ . This is indeed the case as discussed in the next section.

It was proved using different methods [66] that the number of vertices of degree  $k$  is concentrated around its expected value. It is thus natural to expect that  $X_{k,n}$  is asymptotically normal as  $n \rightarrow \infty$ , but this is still an open problem. On the other hand, asymptotic normality of the  $X_{k,n}$  has been established for simpler classes of graphs [26] as well as for planar maps [30].

## 4. Extremal parameters

In this section we focus on several extremal parameters that have been successfully analyzed for random planar graphs. Other extremal parameters will be discussed in the last section.

**Largest component.** Let us start with an easy parameter, the size  $L_n$  of the largest connected component. It has already been mentioned that  $L_n$  is almost

equal to  $n$ , but we can be more precise. Let  $G_n$  be the number of planar graphs and  $C_n$  the number of connected planar graphs. The probability that  $L_n = n - k$ , for fixed  $k$  and  $n > 2k$ , is  $\binom{n}{k} C_{n-k} G_k / G_n$ , since there are  $\binom{n}{k}$  ways of choosing the labels of the vertices not in the largest component,  $C_{n-k}$  ways of choosing the largest component, and  $G_k$  ways of choosing the complement. Using the known estimates for  $G_n$  and  $C_n$  we arrive at

$$\mathbb{P}(L_n = n - k) \sim p \cdot G_k \frac{\gamma^{-k}}{k!}, \quad (13)$$

where  $p$  is the limiting probability of connectivity. Because of (2), this quantity is of order  $k^{-7/2}$  for large  $k$ . It follows that  $n - L_n$  has a limiting discrete distribution with constant expectation and variance. The expected value is computable and  $\mathbb{E}(n - L_n) \approx 0.038$ . Readers familiar with the giant component phenomenon in the  $G(n, M)$  model may wonder about analogs for planar graphs; this will be discussed in the last section.

We can also find the limiting distribution of the *fragment*, the complement of the largest component. The probability that the fragment is isomorphic to a given unlabelled graph  $H$  with  $h$  vertices is, for  $n > h$ ,

$$\binom{n}{h} \frac{h!}{\text{aut}(H)} \frac{C_{n-h}}{G_n},$$

where  $\text{aut}(H)$  is the number of automorphisms of  $H$  and  $h!/\text{aut}(H)$  is the number of different labellings of  $H$ . It follows as before that

$$\mathbb{P}(\text{fragment} \cong H) \sim p \frac{\gamma^{-k}}{\text{aut}(H)}. \quad (14)$$

We will see in Section 6 that this result holds in a more general context.

**Largest block.** Because of the previous result, from now on we focus on *connected* planar graphs. A connected graph decomposes into blocks, which are either single edges (isthmuses) or maximal 2-connected subgraphs. It is natural to consider the size of the largest block. This is a very interesting parameter that has a non-Gaussian continuous limit law. It was first studied for random maps in [38], where it was proved that the largest block in a random map with  $n$  edges has expected size  $\sim n/3$  and, moreover, the second largest block is of order  $O(n^{2/3})$ . This result is somehow comparable to the classical giant component phenomenon, a random map has a unique block of linear size and the other blocks are small. The limiting distribution for the size  $X_n$  of the largest block in random maps was determined very precisely in [3], and it involves the density function  $g(x)$  of a stable law of parameter  $3/2$ . The precise result is the following:

$$\mathbb{P}(X_n = \lfloor n/3 + xn^{2/3} \rfloor) \sim g(x)n^{-2/3}, \quad (15)$$

uniformly for  $x$  in any bounded interval. That is, the largest block has expected size  $n/3$  and fluctuations of order  $O(n^{2/3})$ . It is worth remarking that the distribution

has no second moment and is asymmetric: the left tail (as  $x \rightarrow -\infty$ ) decays polynomially while the right tail (as  $x \rightarrow +\infty$ ) decays exponentially. The proof is based on analyzing the size of the root-block, that is, the block containing the root edge. Equation (5) is the basis of the analysis: the composition scheme  $B(zM(z)^2)$  is *critical*, in the sense that the evaluation of  $zM(z)^2$  at its singularity  $1/12$  is precisely  $4/27$ , which is the singularity of  $B(z)$ . Everything boils down then to estimating the coefficients of large powers of generating functions, which is achieved by a delicate application of the saddle-point method.

Using the tools developed in [3], an analogous result was proved for random planar graphs [44]. In this case the expected size of the largest block (now  $n$  is the number of vertices) is  $\sim \alpha n$ , where  $\alpha \approx 0.96$  (this value of  $\alpha$  was obtained independently in [65] using alternative methods). The limiting distribution is of the same kind as (15), but with a different scaling of  $g(x)$ . The results in [44] also give the limiting distribution for the size of the largest 3-connected component in random connected planar graphs, which again is of the same kind as (15), both in the number of vertices and in the number of edges. The expected number of vertices in the largest 3-connected component is  $\sim 0.73n$ , and the expected number of edges is  $\sim 1.79n$ .

A parameter related to the largest block is the following. The *2-core* of a graph  $G$  is the maximum subgraph  $C$  with minimum degree at least two. The 2-core  $C$  is obtained from  $G$  by repeatedly removing vertices of degree one and, conversely,  $G$  is obtained by attaching rooted trees at the vertices of  $C$ . It is proved in [64] that the size of the 2-core of a random planar graph is asymptotically Gaussian with expectation  $\sim 0.962n$  (the value of the constant was previously found in [57]). The constant is a bit larger than the value 0.96 for the largest block; this is consistent since the 2-core clearly contains the largest block. It is also proved in [64] that the size of the largest tree attached to the 2-core is of order  $c \log n$  where  $c \approx 0.43$ .

**Maximum degree.** Let  $\Delta_n$  be the maximum degree in random planar graphs. A simple and elegant argument by McDiarmid and Reed [58] based on double counting and elementary properties of random planar graphs shows that with high probability

$$c_1 \log n \leq \Delta_n \leq c_2 \log n,$$

for some positive constants  $c_1$  and  $c_2$ . This already gives the right order of magnitude. Analytic methods are needed in order to obtain a more precise result. From the previous section we know that there is a limiting degree distribution  $\{p_k\}_{k \geq 1}$  with tail of order  $q^k k^{-1/2}$ . Using the first moment method and analytic properties of the generating function  $G^\bullet(x, w)$  mentioned in the previous section, one can show that  $\Delta_n \leq (1 + o(1))c \log n$ , where  $c = 1/\log(q^{-1}) \approx 2.53$ . In principle a matching lower bound could be proved using the second moment method, by rooting at a secondary vertex in addition to the root vertex. This is done in [28] for simpler classes of graphs, which is already very demanding. However, the technical difficulties with this approach for planar graphs appear insurmountable, since the equations defining the associated generating functions are just too complicated.

In order to obtain a lower bound one can use *Boltzmann samplers*, introduced in [31] for the random generation of combinatorial objects. If  $\mathcal{A}$  is a class of combinatorial objects with generating function  $A(x)$ , and  $x_0$  is such that  $A(x_0)$  is convergent, then an object  $\alpha \in \mathcal{A}$  of size  $n$  is assigned probability  $x_0^n/A(x_0)$ . The objects generated fluctuate in size, but all the objects of size  $n$  have the same probability.

This framework has been applied successfully since then, in particular to the efficient generation of random planar graphs [36]. One can use Boltzmann samplers not only for random generation but also for the analysis of random combinatorial objects. This approach has proved useful in particular for random planar graphs [65, 66]. This is also the case here, using the fact that there is a unique block of linear size: a typical random planar graph  $G$  can be thought of as a large block  $B$  together with small planar graphs attached to its vertices. If we later condition on the total size of  $G$  being  $n$ , we may start with the graphs attached to  $B$  being drawn independently from the set of all connected planar graphs. In this way one recovers the power of independent samples allowing to use techniques closer to the classical theory of random graphs. This program has been carried out in [29], showing that with high probability

$$|\Delta_n - c \log n| = O(\log \log n),$$

and

$$\mathbb{E}(\Delta_n) = (1 + o(1)) c \log n.$$

**Diameter.** Let  $D_n$  denote the diameter of a random connected planar graph. This is a difficult parameter to analyze, even for relatively simple classes of graphs, such as trees. The starting point is the analysis of metric properties of random planar maps, by now a rich and deep theory with connections to physics and other areas. Let  $Q_n$  be a random embedded quadrangulation (all faces of degree four) with  $n$  faces and let  $r_n$  be the radius (maximum graph distance) in  $Q_n$  with respect to a fixed base point. In the pioneering work [21] it was shown that  $r_n$  is of order  $n^{1/4}$ , in fact, much more was proved:  $r_n/n^{1/4}$  converges in law to a continuous distribution related to Brownian motion. Notice that the diameter of  $Q_n$  is between  $r_n$  and  $2r_n$ . The proof in [21] is based on a bijection between quadrangulations and plane trees enriched with labels that keep track of the distances in  $Q_n$ . The typical height of a tree is of order  $\sqrt{n}$ , and the labels behave like a random walk along the branches of the tree. This implies that the maximum distance is of order  $(\sqrt{n})^{1/2}$ , explaining the exponent  $1/4$ . These results were later extended to other classes of random maps and, more recently, even deeper results have been established. If one consider  $Q_n$  as a metric space with the graph distance  $d_n$ , then  $(Q_n, d_n n^{-1/4})$  converges in a precise technical sense to a certain random compact metric space, known as the Brownian map [52, 61].

One can use the former results to analyze the diameter  $D_n$  in random planar graphs. This is done in [17] starting from the result on quadrangulations and then moving to maps with increasing connectivity. Once a result is established for 3-connected maps, it can be transferred to 3-connected planar graphs and then to

connected planar graphs. One uses in an essential way the existence of a giant block and 3-connected component, both in maps and graphs. The price to pay in this scheme for transferring the results from maps to graphs is a loss in precision. The result proved in [17] is that for  $\epsilon > 0$  small enough and  $n$  large enough,

$$\mathbb{P}(D_n \in (n^{1/4-\epsilon}, n^{1/4+\epsilon})) \geq 1 - \exp(-n^{c\epsilon}).$$

It is natural to conjecture that the radius  $r_n$  of connected planar graphs scaled by  $n^{-1/4}$  converges to the same law as for quadrangulations and other classes of maps, but much more precise results are needed in order to prove such a statement.

**Summary of results.** From this and the previous section we can conclude that we have now a rather complete picture of ‘what a random planar graph looks like’. We summarize the main properties in the following list. All the results are understood to hold asymptotically almost surely when  $n \rightarrow \infty$ . All the constants are explicit and computable to any desired precision. The values given are approximations.

1. The *number of edges* is Gaussian with expectation  $2.21n$  and linear variance.
2. The *number of connected components* is  $1 + \text{Po}(0.037)$ . The probability of being connected is 0.96.
3. If  $L_n$  denotes the *size of the largest component*, then  $n - L_n$  follows a discrete law. The expected value of  $n - L_n$  is 0.38.
4. For each fixed connected planar graph  $H$  rooted at a distinguished vertex, the number of pendant copies of  $H$  is Gaussian with expectation  $(\gamma^{-h}/h!)n$  and linear variance.
5. The chromatic number is four. This follows from the Four Colour Theorem and the fact that it contains  $K_4$  as a subgraph.
6. The *number of automorphisms* is exponential in  $n$ .
7. The *number of blocks* is Gaussian with expectation  $0.039n$ . The *number of cut vertices* is Gaussian with expectation  $0.038n$ . In both cases the variance is linear.
8. For each fixed 2-connected planar graph  $L$ , the *number of blocks isomorphic to  $L$*  is Gaussian with linear expectation and variance.
9. For each  $k \geq 1$ , the expected number of *vertices of degree  $k$*  is  $p_k n$ , where the  $p_k$  are computable and  $\sum p_k = 1$ .
10. The *maximum degree* satisfies  $|\Delta_n - c \log n| = O(\log \log n)$ , where  $c = 2.53$ , and  $\mathbb{E}\Delta_n \sim c \log n$ .
11. The size of the *largest block* has expected value  $0.96n$  and follows a stable law of parameter  $3/2$ . The remaining blocks are of size  $O(n^{2/3})$ . The same holds for the size of the largest 3-connected component, with expectation  $0.73n$ .

12. The size of the  $2$ -core is Gaussian with expectation  $0.962n$  and linear variance.
13. The *diameter*  $D_n$  is in  $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$  with high probability.

## 5. Graphs on surfaces

The theory of map enumeration extends to maps on surfaces. A map on a surface  $S$  is a 2-cell embedding (all faces must be homeomorphic to disks) of a connected graph in  $S$ . It is worth remarking that a map on an orientable surface can be encoded in a purely combinatorial way by means of a rotation system, which consists of giving a cyclic ordering of the edges around each vertex. By giving appropriate signs to the edges the encoding also works for non-orientable surfaces, but for conciseness we only discuss the orientable case [62]. Let  $M_n^g$  be the number of maps with  $n$  edges on the orientable surface of genus  $g$ . As opposed to the planar case, there is no closed formula for  $M_n^g$ , but one can use Tutte's methodology of removing the root edge to find the associated generating function  $M^g(z)$ . Using induction on the genus, it was proved by Bender and Canfield [4] that  $M^g(z)$  is a rational function in  $\sqrt{1-12z}$ . The explicit expression is quite involved but it can be used to prove the estimate

$$M_n^g \sim c_g n^{5(g-1)/2} 12^n. \quad (16)$$

Notice that the genus only affects the subexponential term and not the exponential growth. The surprising exponent  $5(g-1)/2$  was later explained more combinatorially in [20].

Suppose one wishes, as for planar graphs, to use the enumeration of maps on  $S$  for counting graphs (without an embedding) on  $S$ . There are two main obstacles for this program: 1) no degree of connectivity guarantees a unique embedding, and 2) the class of graphs embeddable in  $S$  is not close under taking connected components or blocks, so that the basic equations among generating functions, such as (12) no longer hold. The road to the solution, found independently in [18] and [7], is the following. The *face-width* of a map  $M$  in  $S$  is the minimum number of intersections of  $M$  with a simple non-contractible curve  $C$  on  $S$ . It is easy to see that this minimum is achieved when  $C$  meets  $M$  only at vertices. Face-width is in some sense a measure of local planarity, if the face-width is large then the embedding is locally planar in large balls. The face-width of a graph  $G$  is the maximum face-width among all the embeddings of  $G$ .

The key result is that a 3-connected graph with large enough face-width has a unique embedding [62]. It turns out that the generating series of 3-connected maps of any fixed face-width has a negligible contribution in the asymptotic analysis [8]. Therefore, the enumeration of 3-connected graphs in a surface  $S$  can be reduced, up to negligible terms, to the enumeration of 3-connected maps in  $S$ . There is one technical difficulty, which is to enumerate maps according to edges and a suitable weight on the vertices. This is achieved starting with the enumeration of all maps

in  $S$  and then, using Tutte's approach based on substitution, going to 2-connected and then to 3-connected maps in  $S$ . It is important to remark that, since maps with small face-width are discarded, one does not work with the exact counting series. Instead, if  $f(x)$  is the series of interest, one finds computable series  $f_1(x)$  and  $f_2(x)$  such that  $f_1(x) \preceq f(x) \preceq f_2(x)$  (where  $\preceq$  means coefficient-wise inequality) and  $f_1(x)$  and  $f_2(x)$  have the same leading asymptotic estimates.

For the second obstacle one can use a result from [69]: if a connected graph  $G$  of genus  $g$  has face-width at least two, then  $G$  has a unique block of genus  $g$  and the remaining blocks are planar. A similar result holds for 2-connected graphs and 3-connected components. Since for planar graphs we have exact expressions for all the generating functions involved, starting from the (asymptotic) enumeration of 3-connected graphs of genus  $g$  we can achieve the enumeration of all graphs of genus  $g$ . Let us make more precise one of the steps in the analysis. Let  $G^g(x)$  and  $C^g(x)$  be the generating functions of graphs and connected graphs of genus at most  $g$ , respectively. The usual relation  $G^g(x) = \exp(C^g(x))$  does not hold, since the union of graphs of genus  $g$  will have larger genus if  $g > 0$ . Instead, we have

$$G^g(x) \sim C^g(x)e^{C^0(x)},$$

where the symbol  $\sim$  must be understood as the fact that the two functions have the same dominant terms in their singular expansions. Similarly, the relation between  $C^g(x)$  and the generating function  $B^g(x)$  of 2-connected graphs of genus  $g$  is not an exact equation as in the planar case, since genus is also additive in blocks, but rather an approximate version. The technical details are involved but the essence is to discard maps and graphs with small face-width.

In addition, the former approach allows one to analyze parameters of a random graph embeddable in the surface  $S^g$  of genus  $g$ . All the main parameters behave as in the planar case: number of edges is Gaussian with the same moments, number of components is 1 plus a Poisson law with the same parameter, the size of the largest component follows the same law as in (13), and the size of the largest 2-connected and 3-connected components obey stable laws with the same expectations. In addition, a random graph embeddable in  $S^g$  almost surely does not embed in a simpler surface. Thus we have a clear picture of what a random graph embeddable in  $S^g$  looks like. It has a unique largest component  $C$  of genus  $g$  and the remaining components are planar. Within  $C$  there is a unique block  $B$  of linear size that has genus  $g$  and the remaining blocks are planar. Finally,  $B$  has a unique linear 3-connected component  $T$  of genus  $g$ , and the remaining 3-connected components are planar. Moreover, the graph  $T$  has a unique embedding in  $S^g$ . Extremal parameters like the diameter or the maximum degree behave likely as in the planar case, but the analysis is yet to be done.

We conclude this section with a short comment. Given a connected planar graph  $H$ , a random graph in  $S^g$  contains linearly many pendant copies of  $H$ , the proof being the same as for planar graphs. But if  $H$  is non-planar then a random graph in  $S^g$  does not contain  $H$  as a subgraph almost surely, because all the balls of radius  $R$  are planar for each fixed  $R$ . Taking  $R$  larger than the diameter of  $H$  we would reach a contradiction.

## 6. Minor-closed classes of graphs

We recall that a class of graphs  $\mathcal{G}$  is minor-closed if whenever  $G$  is in  $\mathcal{G}$  and  $H$  is a minor of  $G$ , then  $H$  is also in  $\mathcal{G}$ . The theory of graph minors is one of the main achievements in modern combinatorics, culminating with the great theorem of Robertson and Seymour: every minor-closed class of graphs is defined in terms of a *finite* number of excluded minors; see [53] for a quick overview. The basic example is Kuratowski's theorem, which identifies  $K_5$  and  $K_{3,3}$  as the excluded minors for planar graphs. There are several important properties that have been established for proper (excluding at least one graph) minor-closed classes of graphs. To begin with they are sparse: the number of edges is at most  $\alpha n$  for some constant  $\alpha$  depending only on the class. This is easy to prove with  $\alpha = 2^t$ , where  $t$  is the size of an excluded minor, although the correct order of magnitude of  $\alpha$  is  $t\sqrt{\log t}$  [72]. Secondly, they are *small*: the number  $G_n$  of graphs in the class with  $n$  vertices is bounded as

$$G_n \leq c^n n!,$$

for some constant  $c > 0$ . This implies in particular that the generating function  $G(x) = \sum G_n x^n / n!$  has positive radius of convergence and defines an analytic function near 0. This was first proved in [63] and then in [32] in a more general context. Additional properties are, for example, the existence of separators of size  $O(\sqrt{n})$  and the fact that the tree-width is  $O(\sqrt{n})$  [2].

The systematic study of *random* graphs from a minor-closed class is more recent. Let  $\mathcal{G}$  be a proper minor-closed class which is *addable*. This means that 1) a graph  $G$  is in  $\mathcal{G}$  if and only the connected components of  $G$  are in  $\mathcal{G}$ ; 2) for each graph  $G$  in  $\mathcal{G}$ , if  $u$  and  $v$  are vertices in different components of  $G$ , the graph obtained by adding an edge joining  $u$  and  $v$  is also in  $\mathcal{G}$ . This is equivalent to the condition that all the excluded minors of  $\mathcal{G}$  are 2-connected. Planar graphs form an addable class, but graphs embeddable in a surface other than the sphere do not, since genus is additive on disjoint unions. Addable minor-closed classes are analyzed by McDiarmid in [56]. The first property, already proved in [60], is the existence of a growth constant  $\gamma$ , which is the limit

$$\gamma = \lim_{n \rightarrow \infty} \left( \frac{G_n}{n!} \right)^{1/n}. \quad (17)$$

In fact, more is true. The class  $\mathcal{G}$  is called *smooth* if

$$\lim_{n \rightarrow \infty} \frac{G_n}{nG_{n-1}} = \gamma. \quad (18)$$

Of course, if the former limit exists it must equal  $\gamma$ , but condition (18) is stronger than (17). It is shown in [56] that addable minor-closed classes are smooth. This is proved using the 2-core discussed before and applying a technique from [5].

From now on  $\mathcal{G}$  is an addable minor-closed class and  $R_n$  is a random graph from  $\mathcal{G}$  with  $n$  vertices under the uniform distribution. Several basic properties have been established for  $R_n$ . It was already proved in [60] that  $R_n$  contains a



linear number of pendant copies of every fixed connected graph  $H$  in  $\mathcal{G}$ . Using the smoothness condition this is strengthened in [56], as follows. If  $X_n$  is the number of pendant copies of  $H$  in  $R_n$ , then

$$\frac{X_n}{n} \rightarrow \frac{\gamma^{-h}}{h!} \quad \text{in probability,} \quad (19)$$

where  $h$  is the number of vertices in  $H$ . In the case of planar graphs a stronger result is shown in [42] using analytic methods, namely that  $X_n$  is asymptotically Gaussian with expectation  $(\gamma^{-h}/h!)n$ . The great interest of the less precise result (19) is that it holds for *every* addable minor-closed class, where generating functions are seldom available. In particular, we deduce that for each  $k \geq 1$  there is a linear number of vertices of degree  $k$ , and that the number of automorphisms is exponential.

The more precise results from [56] are on the structure and number of connected components. Let  $\rho = \gamma^{-1}$ , which is the radius of convergence of the counting generating function  $G(x)$ . We have  $0 < \rho \leq 1/e$ . The first inequality because  $\mathcal{G}$  is small, and the second one because  $\mathcal{G}$  contains the class of forests, which grows exponentially like  $e^n n!$ . It also holds that  $G(\rho)$  is finite [56]. Let now  $\mathcal{C}$  be the set of connected graphs in  $\mathcal{G}$ , and let  $C(x)$  be the associated generating function. From general enumerative principles [35] we have the relation  $G(x) = \exp C(x)$ , and it follows that  $C(\rho)$  is finite too. Denote by  $L_n$  the size of the largest component. We can now describe the main results from [56]. As before, all statements hold asymptotically almost surely. For a given graph  $H$ , we denote the number of vertices by  $|H|$ .

1. The number of components is distributed like  $1 + \text{Po}(C(\rho))$ . In particular, the probability of connectedness is  $e^{-C(\rho)}$ .
2. For distinct unlabelled connected graphs  $H_1, \dots, H_k$  in  $\mathcal{G}$ , the numbers of components  $X_i$  isomorphic to  $H_i$  are asymptotically independent with distribution  $\text{Po}(\lambda_i)$ , with  $\lambda_i = \rho^{|H_i|}/\text{aut}(H_i)$ .
3.  $n - L_n$  follows a discrete law. For each fixed  $k$ ,

$$\mathbb{P}(n - L_n = k) \rightarrow \frac{1}{G(\rho)} \frac{G_k \rho^k}{k!}.$$

4. Given a fixed graph  $H$ , the probability that the fragment (the complement of the largest component) is isomorphic to  $H$  tends to

$$\frac{\rho^{|H|}}{\text{aut}(H) G(\rho)}.$$

Notice that item 3 corresponds exactly to equation (13), since  $\rho = \gamma^{-1}$  and  $1/G(\rho) = e^{-C(\rho)}$  is the probability of being connected. The same applies to item 4 with respect to (14). Which values are possible for the limiting probability of connectivity  $e^{-C(\rho)}$ ? It was conjectured in [60] that, among all addable classes, this

probability is minimized for the class of forests, in which case it is  $e^{-1/2}$ . This conjecture has been proved independently in [1] and [48].

For other parameters of interest, like the number of edges, there are no general results available. The number of edges is linear by the general bound on minor-closed classes, but we do not know how to prove, for instance, any concentration result. The same goes for the number of vertices of given degree and other basic parameters. Adapting the techniques from [58], it is proved in [41] that for addable classes whose excluded minors are all 3-connected, the maximum degree  $\Delta_n$  is at least  $c \log n$  for some constant  $c > 0$  (this does not apply, for instance, to the class of forests, where  $\Delta_n \sim \log n / \log \log n$ ). For any addable minor-closed class it is conjectured that  $\Delta_n \leq c' \log n$ , but the proof of the upper bound for planar graphs in [58] does not extend to the general case.

For non-addable classes there are few general results, but some very interesting examples. Let  $\mathcal{G}_k$  be the class of graphs containing at most  $k$  disjoint cycles. This class is minor-closed but not addable. Let  $\mathcal{F}_k$  be the class of graphs  $G$  such that removing  $k$  vertices from  $G$  the graph becomes a forest. In other words, graphs in  $\mathcal{F}_k$  are obtained from a forest  $F$  by adding  $k$  new vertices and connecting them in any way to  $F$ . Clearly  $\mathcal{F}_k \subset \mathcal{G}_k$ . It is proved in [50] that almost every graph in  $\mathcal{G}_k$  is in  $\mathcal{F}_k$ , as  $n \rightarrow \infty$ . This gives in particular the asymptotic growth of  $\mathcal{G}_k$ , since it can be shown that the number of graphs in  $\mathcal{F}_k$  grows like

$$c_k 2^{kn} f_n,$$

where  $f_n$  is the number of forests and  $c_k$  is an explicit constant. The simple structure of graphs in  $\mathcal{F}_k$  also gives access to properties of random graphs from  $\mathcal{G}_k$ . This approach has been generalized to other classes excluding disjoint copies of a given family of graphs [51].

Another example of a non-addable class is the class  $\mathcal{A}$  of graphs whose components are caterpillars; a caterpillar is a tree obtained from a path by adding leaves. This class and related classes can be analyzed using generating functions [16]. It is proved, for instance, that the number of components in  $\mathcal{A}$  follows a Gaussian law with expectation of order  $\sqrt{n}$ , a very different behaviour from what we have seen in addable classes.

To conclude this section, we mention a recent result on logical limit laws [45]. Consider a graph property expressible in first order (FO) logic, for example the existence of a triangle or the existence of an isolated vertex. Given a class of graphs  $\mathcal{G}$ , we are interested in the limiting probability  $p(\phi)$ , as  $n \rightarrow \infty$ , that a FO formula  $\phi$  is satisfied in  $\mathcal{G}$ , provided this limit exists. This problem has been much studied for the random graph  $G(n, p)$ . One of the earliest results is that for constant  $p$  (in particular  $p = 1/2$ , the uniform model on labelled graphs), for every first order property  $\phi$  we have either  $p(\phi) = 0$  or  $p(\phi) = 1$ . This is called a zero-one law (see [71] for much more in this area). Zero-one laws have been studied for other combinatorial structures, such as permutations or partitions [22], and also for maps on surfaces [6]. More recently, a zero-one law was proved for random labelled trees [54]. Moreover, it holds for properties expressible in the richer monadic second order (MSO) logic, in which we are allowed to quantify

over sets of vertices, in addition to quantifying over vertices. Properties such as connectivity or  $k$ -colorability can be expressed in MSO but not in FO. It is proved in [45] that for every addable minor-closed class  $\mathcal{G}$  and every MSO formula  $\phi$ , the limiting probability  $p(\phi)$  exists. Moreover, if we restrict to connected graphs in  $\mathcal{G}$ , then a zero-one law holds. It is also proved that the closure of the set  $\{p(\phi) \mid \phi \text{ MSO formula}\}$  of limiting probabilities is a finite union of at least two intervals in  $[0, 1]$ . For the class of planar graphs, the set of intervals is completely determined.

## 7. Subcritical classes

For the next definition we need a bit more on generating functions. Let  $\mathcal{G}$  be a class of graphs which is *block-stable*, that is, a graph  $G$  is in  $\mathcal{G}$  if and only if each of the blocks of  $G$  is in  $\mathcal{G}$ . This is the case, for instance, for addable minor-closed classes defined in the previous section. In this situation, as we saw in Section 2, the generating functions  $C(x)$  and  $B(x)$  of connected and 2-connected graphs in  $\mathcal{G}$  satisfy

$$C^\bullet(x) = xe^{B'(C^\bullet(x))}, \quad (20)$$

where  $C^\bullet(x) = xC'(x)$  is the generating function of connected graphs rooted at a vertex. Let  $\rho_C$  and  $\rho_B$  be, respectively, the radius of convergence of  $C(x)$  and  $B(x)$ . We say that  $\mathcal{G}$  is subcritical if

$$C^\bullet(\rho) < \rho_B.$$

This implies that the singular behaviour of  $C(x)$  is dictated by the existence of a critical point when solving (20), and not by the singular behaviour of  $B(x)$  at  $\rho_B$ . In fact, the critical point is the solution of  $xB''(x) = 1$ . The class of planar graphs is *critical*, since in this case  $C^\bullet(\rho) = \rho_B$ . It is clear that this is a delicate condition, since it depends on whether a certain evaluation of an analytic function is smaller than or equal than another value. Unless we have access to the generating functions, it seems that we cannot prove whether a given class is subcritical or not.

A basic example of a subcritical class is the class of series-parallel graphs; they can be characterized in several ways, among them as the graphs not containing  $K_4$  as a minor. This class is subcritical, as shown first in [11]. Other examples are outerplanar graphs, acyclic graphs (forests), and cacti graphs (graphs whose blocks are cycles). As shown in [44], the class of graphs not containing  $H$  as a minor is subcritical in several other cases, including  $H = K_5 - e$  (the complete graph  $K_5$  minus an edge). A general framework was introduced in [44] for analyzing block-stable classes of graphs whose 3-connected components are predefined. A fundamental dichotomy was found (see also [65]) between critical and subcritical classes. As we have seen, a random planar graph has a block of linear size. In contrast to this, a random graph from a subcritical class has blocks of size  $O(\log n)$  and the block size follows a discrete distribution. In a sense, subcritical classes are close to trees: a typical graph is made of a linear number of small blocks forming a tree whose height is of order  $\sqrt{n}$ . We discuss further this dichotomy in the last section.

With respect to other parameters such as the number of edges or the number of components, the behaviour is the same for critical and subcritical classes. It is worth remarking that the only examples we know of critical classes are planar graphs and classes very close to them, such as graphs not containing  $\mathcal{K}_{3,3}$  as a minor [44]. A systematic study of subcritical classes was done in [25]. It is shown that the asymptotic growth is always of the form  $c \cdot n^{-5/2} \gamma^n n!$  for computable constants  $c$  and  $\gamma$ . Remarkably, this is also proved for the corresponding *unlabelled* classes, where symmetries have to be taken into account and cycle-index sums are needed; the estimate in this case is of the form  $c_u n^{-5/2} \gamma_u^n$ . In addition, the number of edges and other linear parameters are asymptotically Gaussian with linear expectation and variance.

## 8. Concluding remarks

In this section we discuss additional aspects of random planar graphs and related classes, and several open problems. So far we have discussed random planar graphs according to the number of vertices, but it is also interesting to consider the number  $G_{n,m}$  of planar graphs with  $n$  vertices and  $m$  edges. There are two different situations. First, when  $m = \alpha n$  for  $\alpha \in (1, 3)$ . This was addressed in [42], and it was shown that

$$G_{n, \lfloor \alpha n \rfloor} \sim c(\alpha) n^{-4} \gamma(\alpha)^n n!,$$

where  $c(\alpha)$  and  $\gamma(\alpha)$  are analytic functions of  $\alpha$ . The function  $\gamma(\alpha)$  has a strict maximum at  $\mu \approx 2.21$ , where  $\mu n$  is the expected number of edges. This proves in particular the large deviations result for the number of edges. It turns out that the typical behaviour of random graphs with  $\alpha n$  edges is qualitatively the same for each  $\alpha \in (1, 3)$ , that is, there is no critical value of  $\alpha$ . The matter changes if one considers  $m \leq n$ . As shown in [47], there are two critical periods in the ‘evolution’ of planar graphs with  $n$  vertices and  $m$  vertices. The first one is analogous to the phase transition observed in the standard  $G(n, M)$  model and takes place for  $M = n/2 + O(n^{2/3})$ , when the largest complex component is formed. A second critical period appears at  $n + O(n^{3/5})$ , when the complex components cover nearly all vertices.

So far we have worked with labelled graphs, but all our problems make sense for unlabelled graphs as well. Let  $U_n$  be the number of unlabelled planar graphs with  $n$  vertices. We do not have yet a precise estimate for  $U_n$ , we do not even know the unlabelled growth constant  $\gamma_u = \lim(U_n)^{1/n}$ . Since the number of automorphisms of a random labelled planar graph is exponential, we must have  $\gamma_u > \gamma = 27.23$ . On the other hand, the best upper bound available is  $\gamma_u < 30.06$ , proved in [15]. Using Pólya’s theory of counting, unlabelled graphs can be enumerated for subcritical classes [25]. In principle this could be doable for planar graphs starting at 3-connected planar graphs, but the analysis of symmetries appears too involved. In any case, one should expect an asymptotic estimate of the form  $U_n \sim c n^{-7/2} (\gamma_u)^n$ . Also, random unlabelled planar graphs should share the same properties as their labelled counterpart.

Another open problem we address is the possible dichotomy discussed in the previous section between critical and subcritical classes. Let  $\text{Ex}(H_1, \dots, H_k)$  be the class of graphs not containing any of the  $H_i$  as a minor. For instance,  $\text{Ex}(K_5, K_{3,3})$  is the class of planar graphs and  $\text{Ex}(K_4)$  is the class of series-parallel graphs. In all cases where analytic methods are available, one observes that the class is subcritical if and only if at least one of the excluded minors is planar. A central result in the graph minors program [68], says that the tree-width of graphs in  $\text{Ex}(H_1, \dots, H_k)$  is bounded if and only if at least one of the  $H_i$  is planar. The tree-width is a measure of how close is a graph to being tree-like. If we recall that graphs from subcritical classes are typically tree-like, the following conjecture seems reasonable, restricted to addable classes, where the basic equation (20) holds.

*Conjecture.* The class  $\text{Ex}(H_1, \dots, H_k)$  is subcritical if and only if at least one of the  $H_i$  is planar, which is equivalent to having bounded tree-width.

In particular, it would be very interesting to prove this conjecture for the class  $\mathcal{G}_k$  of graphs with tree-width at most  $k$ .  $\mathcal{G}_1$  is the class of forests and  $\mathcal{G}_2$  is the class of series-parallel graphs, which are subcritical. But already for  $\mathcal{G}_3$  we do not know. We know which are the edge-maximal graphs in  $\mathcal{G}_k$ , the so-called  $k$ -trees. They certainly have a tree-like structure (almost by definition) but it is not clear how to infer results for random graphs in  $\mathcal{G}_k$  from the maximal ones.

Another topic for future research is to analyze additional extremal parameters. The following questions refer to almost sure properties of random planar graphs.

- *Cores.* The  $k$ -core of a graph is the maximum subgraph with minimum degree at least  $k$ . We have already discussed the 2-core, which is of linear size for random planar graphs. The 3-core is not necessarily connected, but it is conjectured [64] that the 3-core contains a component of linear size, and that the components of the 4-core are all sublinear.
- *Tree-width.* It is known that a planar graph with diameter  $D$  has tree-width  $O(D)$ . It follows that the tree-width is at most  $O(n^{1/4+\epsilon})$ . Is this the right order of magnitude? We remark that there are planar graphs (grids) with tree-width  $\sqrt{n}$ .
- *Longest cycle.* We conjecture the existence of cycle of length  $cn$  for some  $c > 0$ . Because of the results on the largest 3-connected component, it would be enough to prove it for random 3-connected planar graphs. In contrast, it is easy to see that there is always a matching of linear size (consider pendant copies of a single edge).

On the enumerative side, we mention the problem of counting 4-regular planar graphs. Cubic planar graphs can be enumerated adapting Tutte's decomposition into 3-connected components [12], but this approach does not seem to work for higher degree. In the same way, planar graphs with minimum degree three can be enumerated [64], but the same obstacle appears for minimum degree four. Another problem is to enumerate bipartite planar graphs. The real difficulty is to keep

control of the bipartite character in the decomposition of 2-connected graphs into 3-connected components.

Concerning minor-closed classes of graphs, a main open problem is to show that the growth constant always exists (as conjectured in [10]). More of a metaproblem is to analyze additive parameters like the number of edges or extremal parameters like the size of the largest block in general minor-closed classes. It is not at all clear that there is a way of attacking them without precise enumerative results. One case particularly appealing is the class  $\text{Ex}(K_5)$ . Wagner's theorem tells us how is the structure of graphs in  $\text{Ex}(K_5)$ , but so far we are not able to obtain precise enumerative information from it.

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